Exercise 7

In Exercises 5-8, derive the general solution of the given equation by using an appropriate change of variables, as we did in Example 3.

$$\frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} = 2.$$

Solution

Make the change of variables, $\alpha = x + 2t$ and $\beta = x - 2t$, and use the chain rule to write the derivatives in terms of these new variables.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha}(1) + \frac{\partial u}{\partial \beta}(1) = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = \frac{\partial u}{\partial \alpha}(2) + \frac{\partial u}{\partial \beta}(-2) = 2\frac{\partial u}{\partial \alpha} - 2\frac{\partial u}{\partial \beta}$$

The PDE then becomes

$$\begin{aligned} 2 &= \frac{\partial u}{\partial t} - 2\frac{\partial u}{\partial x} \\ &= \left(2\frac{\partial u}{\partial \alpha} - 2\frac{\partial u}{\partial \beta}\right) - 2\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) \\ &= -4\frac{\partial u}{\partial \beta}. \end{aligned}$$

Divide both sides by -4.

$$\frac{\partial u}{\partial \beta} = -\frac{1}{2}$$

Integrate both sides partially with respect to β to get u.

$$u(\alpha, \beta) = -\frac{1}{2}\beta + f(\alpha)$$

Here f is an arbitrary function. Now that the general solution to the PDE is known, change back to the original variables.

$$u(x, t) = -\frac{1}{2}(x - 2t) + f(x + 2t)$$